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A NEW TECHNIQUE FOR HIGH LEVEL DECISION SUPPORT

by

E.J. EMOND
D.W. MASON

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E.J. Emond
D.W. Mason

Recommended by: _____ DOR(CAM)

Approved by: ABradfield DGOR

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ABSTRACT

The Operational Research Division of the Canadian Department of National Defence receives frequent requests for high level decision support, many of which turn out to be variants of the weighted *m*-rankings problem. This report presents a new approach to the problem of finding a solution ranking which has maximum agreement with the input rankings. The input rankings may be weighted, may contain unlimited ties and may be incomplete. We have developed a new rank correlation coefficient which we have called *tau-x* because it is an extension of Kendall's *tau*. *Tau-x* is ideal as a measure of agreement in the weighted *m*-rankings problem. We have developed a branch and bound algorithm to search the solution space and a heuristic algorithm to use in large problems. Our methods are being successfully applied to support high level decision making in the Canadian Department of National Defence.

RÉSUMÉ

La Division de la recherche opérationnelle du ministère canadien de la Défense nationale reçoit fréquemment des demandes d'aide pour la prise de décisions de haut niveau, dont bon nombre sont en réalité des variantes du problème de classement-*m* pondéré. Le présent rapport présente une nouvelle façon d'aborder le problème en vue de trouver un classement des solutions qui corresponde le mieux aux classements des entrées. Les classements des entrées peuvent être pondérés, peuvent contenir des liens illimités et peuvent être incomplets. Nous avons mis au point un nouveau coefficient de corrélation du classement que nous avons appelé *tau-x* parce qu'il est une extension du *tau* de Kendall. Le *tau-x* est idéal comme mesure de concordance dans le cas du problème des classements-*m* pondérés. Nous avons conçu un algorithme de séparation et d'évaluation pour effectuer une recherche de l'espace de solution et un algorithme heuristique qui servira dans le cas des gros problèmes. Nos méthodes sont utilisées avec succès pour appuyer la prise de décisions de haut niveau au sein du ministère canadien de la Défense nationale.

EXECUTIVE SUMMARY

The weighted *m*-rankings problem occurs frequently in the Department of National Defence. Examples include committees such as the Army Equipment Board which creates a prioritized list of land force equipment projects; classical multi-criteria decision problems where contending options are ranked based on weighted criteria; and complex mixes of the two such as a recent exercise to determine departmental priorities for capital equipment spending. In each case the goal was to provide a ranking of competing objects based on a set of input rankings.

The basic concept employed in this report is to find a ranking which agrees most closely with the set of input rankings, where agreement is measured by a rank correlation coefficient. The solution space is the set of all rankings of *n* objects where ties are allowed. The solution will be a ranking which gives the maximum weighted average correlation with the input rankings.

The two best known rank correlation coefficients were tried and ultimately rejected as measures of agreement for the weighted *m*-rankings problem. Kendall's *tau* was found to exhibit unacceptable properties because of the way it handles ties. Spearman's *rho* suffers from sensitivity to irrelevant alternatives. A new rank correlation coefficient based on Kendall's *tau* was developed and found to have all the compelling and logically sound properties of *tau* but handled ties without the logical flaws. The new rank correlation coefficient has been named *tau-x* because it is an extension of Kendall's *tau*. We have given a formal development of *tau-x* based on the concept of interchanges or flips, showing how it arises naturally if the concept of the half-flip is introduced to create or break ties. *Tau-x* has been found to be an ideal measure of agreement in the weighted *m*-rankings problem with the ability to handle ties, weights and incomplete rankings in a sound and logical way.

Our solution concept for the weighted *m*-rankings problem requires searching the space of all possible rankings of *n* objects with ties allowed. This space is huge, growing as the product of an exponential and a factorial. We have developed a branch and bound algorithm which has been applied successfully in problems with moderate consensus

involving up to about 30 ranked objects. For larger problems we have developed a heuristic algorithm which is capable of handling problems with hundreds of ranked objects. The algorithm entails the iterated application of the branch and bound procedure over a sliding window containing a subset of the objects.

The methodology and techniques we have developed for the weighted m-rankings problem are currently being applied successfully in the Department of National Defence. Our methods provide a mathematically sound and practical approach to the variety of ranking problems which occur so frequently in high level decision support.

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A NEW TECHNIQUE FOR HIGH LEVEL DECISION SUPPORT

I. INTRODUCTION

The Weighted M-Rankings Problem

1. Over the last 20 years the Operational Research Division of the Canadian Department of National Defence has received many requests for analytical assistance from our clients on decision problems which have turned out to be variants of the weighted *m*-rankings problem. In this type of decision problem we are given *m* rankings of *n* objects. The rankings may contain ties of any order and may or may not be complete in the sense that one or more objects may be missing from a ranking. Each of the *m* rankings has a positive weight which represents its relative importance in determining the solution or consensus ranking. The solution is a complete ranking of the *n* objects which best agrees with the input rankings.

2. In cases where the decision problem involved finding the consensus of rankings from equally weighted committee members, we have been successful in applying a locally developed decision tool, the Consensus Decision Support Program, known by its initials, CDSP. This program finds a consensus ranking from equally weighted input rankings using coalition analysis based on *n*-person game theory. In more complicated cases involving weighted rankings we have been unable to use CDSP and have had to improvise, often with methods which were not ideal. There have also been cases such as those with partial rankings in which, until now, we have had no suitable analytical tools.

3. Our recent research on this problem has led to the development of a new set of analytical techniques which promise to provide a practical and theoretically sound methodology for all the variations of the weighted *m*-rankings problem. The new methods are based on a solution concept which has great intuitive appeal and which provides a general approach to a whole family of decision problems. Along the way we have created a new measure of rank correlation which we have called *tau-x* since it is a logical extension of Kendall's *tau* rank correlation coefficient. *Tau-x* has all of the

appealing properties of Kendall's *tau* but does not have the problems from which *tau* suffers.

4. In this paper, we will first describe some typical examples of the kinds of decision problems which we have encountered in the past. These will serve to illustrate the techniques to be described in detail in subsequent sections. The examples also will give the reader an idea of the types of problems with which we are concerned. A mathematical description of the weighted *m*-rankings problem will be given next, followed by a discussion of the solution concept and the problems we have found with the most common rank correlation coefficients including Kendall's *tau* and Spearman's *rho*. We will conclude with a detailed example and some practical observations which may be helpful to analysts confronted with this type of problem.

Defence Oriented Examples

5. **Example 1 - The Army Equipment Board**. The Army Equipment Board is a committee of representatives from various branches and specialties of the Canadian Armed Forces. This committee has the yearly task of determining a prioritized listing of Land Force equipment acquisition projects. Given a detailed description of up to 130 equipment goals, each of up to 13 board members creates a complete ranking based on his or her perception of the urgency of the equipment goals and the relative contribution of each to the achievement of the priorities of the Canadian Army. The board members are given equal weight. The analytical task is to create a consensus ranking which can be used as a guide by senior management to allot funding for the individual projects and help to create a long term equipment acquisition plan.

6. **Example 2 - Transport Aircraft Color Selection**. This example represents a common decision situation - the classical multi-criteria decision problem. In contrast to Example 1 where we had input rankings from committee members, here we have rankings of contending options based on weighted criteria. In our particular example there were five camouflage patterns which were deemed acceptable for a fleet of tactical transport aircraft. Eight ranking criteria were agreed upon and the analyst was provided with an evaluation of each camouflage pattern with respect to each of the criteria.

Criteria weights were developed by assigning the least important criterion a value of 1 and the most important criterion a value of 10. All the others were given integer weights between these values. The analytical problem was to combine the input rankings to produce an overall solution ranking, taking into account the information contained in the rankings and the relative weights in a logical way.

7. **Example 3 - Strategic Priorities for National Defence**. The final example is a case that occurred in a recent high level exercise to determine strategic priorities for future capital spending in the Canadian Department of National Defence. This example combines the features of the first two making a challenging case for our analytical methods. In the exercise the various branches of National Defence were represented on a working committee. In a series of meetings over several weeks the representatives from the branches considered the relative desirability of 31 departmental initiatives according to seven fundamental criteria. After several iterations the group produced 42 separate rankings of the 31 projects, representing assessments by individual committee members on behalf of the various criteria. The working committee was also tasked to create weights for the rankings. The challenge for the analysts was to determine a single ranking of the 31 projects which could be used by senior management to guide departmental priorities.

8. As can be seen from the three examples discussed above, there are common features to the type of problem with which we are dealing. In every case there is a requirement to provide a solution ranking of n objects based on m input rankings of the same objects.

Notational Conventions

9. There are several equivalent ways in which rankings can be specified. One way is to list the rank values of the objects starting with the rank of the first object. This method may be convenient computationally but can sometimes be confusing in a written report. Another way is to list the objects themselves in rank order using the integers from 1 to n as labels. In this paper we will use the second method and we will indicate that we are doing so by enclosing the object labels in angled brackets. Ties are indicated by minus

signs. An object is tied with the one immediately preceding it in a ranking if it is written with a minus sign. Ties of any order can be handled using this convention. For example, the following ranking of six objects has object 5 ranked first, objects 1, 3 and 6 tied for second, followed by objects 4 and 2, respectively: < 5 1 –3 –6 4 2 >.

Discussion of Ties and Weights

10. **The Meaning of Ties**. The concept of ties in ranking data seems at first to be clear and simple, but this apparent happy situation is an illusion. Ties can be very troublesome and have been the source of much confusion in the past.

11. If the ranking has resulted from measuring or otherwise evaluating the desirability of the objects according to a specific criterion, such as cost, then a tie between two or more objects simply means that their relative desirability according to the criterion in question has been found to be the same.

12. If the ranking represents the opinion of a human being (or group of human beings), then a tie between objects can have different interpretations. It may be that the ranker believes that the tied objects are equivalent in the sense that he or she is indifferent to their relative order in the solution. This indifference could be interpreted as a desire to have the other rankers determine the relative order of the tied objects. Alternatively it may be that the ranker believes that the objects are equivalent in a stronger sense and that none of the tied objects should be preferred over the other. In this case the ranker's intention is to have the objects tied in the solution ranking.

13. Ties also present some interesting questions when we consider what is required as a solution in the *m*-rankings problem. To simplify the discussion, we will consider a situation in which only two objects are involved. Suppose we have three equally weighted rankings of two objects as follows.

Ranking 1. < 1 -2 >
Ranking 2. < 1 -2 >
Ranking 3. < 1 2 >

14. In this simple case the first two rankings have the objects tied and the third ranking has object 1 ahead of object 2. If we require a solution ranking which most closely agrees with the three input rankings, the tied ranking $< 1\ -2 >$ is the correct choice since it is identical to two of the three input rankings. However, if we need to determine a winner in the sense of choosing either object 1 or 2, then object 1 would be chosen because it is preferred over object 2 in one ranking and tied in the two others. The correct solution in this case would be the ranking $< 1\ 2 >$. This example illustrates that there can be different answers to the same problem, both correct in different circumstances. The analyst must be aware of this and take the time to consider the final use to be made of the solution to any given *m*-rankings problem.

15. As a final point we note that even where the problem requires that a winner be chosen, a tie may still be the most reasonable solution. This would be the case for example if the input rankings display symmetry of preference for all objects. For this reason it may be necessary to allow ties in the solution ranking, even if none are allowed in the input rankings.

16. **The Meaning of Weights**. Before proceeding with a mathematical discussion of our solution concept for the weighted *m*-rankings problem, we will briefly discuss what is meant by weighting the input rankings. Intuitively, assigning weights to each of the *m* rankings conveys the information that those rankings with larger weight should be given more influence in determining the solution. Exactly how we deal with this requirement will depend on the solution concept employed.

17. We wish to find a solution ranking which simultaneously agrees most closely with the input rankings. Agreement will be measured by a rank correlation coefficient which we will use as a metric in the sense that it gives the relative 'distance' between any two rankings. The solution will be a ranking which gives the smallest average distance from the *m* input rankings. In the case where we have weighted rankings, we will use a weighted average and this will automatically drive the solution towards greater agreement with the most heavily weighted rankings, exactly as one would intuitively wish.

Solution Concept For The M-Rankings Problem

18. In this paper we will deal with only one solution concept, namely finding a solution ranking which in a specific mathematical sense agrees most closely with the set of input rankings. Thus the solution will be a ranked list of the contending objects rather than a single winner. Of course the object(s) ranked first can be considered as the winner(s) if desired.

19. In the basic weighted *m*-rankings problem we are given *m* complete rankings of *n* objects: $R_1, R_2, \ldots, R_m$. Any of the rankings may contain ties and each ranking has a positive weight, w_i. The solution concept is to find a solution ranking R_s which maximizes the following function.

$$\text{Maximize :} \quad \frac{\sum_{i=1}^{m} [w_i * Agreement(R_s, R_i)]}{\sum_{i=1}^{m} w_i} \qquad (1)$$

20. The next step is to find a suitable measure for determining the agreement between two rankings. The obvious measure of agreement is the rank correlation coefficient, Kendall's *tau* and Spearman's *rho* being the two best known examples. Both were tried and ultimately rejected. A full discussion of the problems with Spearman's *rho* and Kendall's *tau* in the *m*-rankings problem will be given below. At this point it is appropriate to introduce a new rank correlation coefficient which we have called *tau-x* since it is similar to Kendall's *tau* but does not suffer from the problems which we have found with *tau*.

II. RANKING SPACE METRICS AND TAU-X

Introduction

21. In his 1948 treatise, Rank Correlation Methods (Reference 1), Kendall introduced *tau* and gave an interpretation of it as a measure of disarray between two rankings. He discussed the concept of interchanges between pairs and pointed out for the no-ties case that any ranking can be transformed into any other ranking by a process of interchanges between pairs of objects. The minimum number of interchanges or flips to get from one ranking to another is a measure of agreement between rankings. Kendall proved that this measure is equivalent (when standardized) to *tau* for the no-ties case. Kendall's *tau* rank correlation coefficient has all the logical properties of a correlation coefficient and is particularly appealing when viewed using the analogy to flips.

22. Kendall extended *tau* to the case where ties are permitted in two different ways, calling the two versions *tau-a* and *tau-b*. Unfortunately neither version is satisfactory as a metric in the space of all rankings of n objects when ties are allowed. In an attempt to derive a more suitable measure of the distance between rankings of n objects when ties are allowed, we have taken the idea of using the minimum number of interchanges or flips to get from one ranking to another and extended it in a logical way to derive *tau-x* which is a new rank correlation coefficient. *Tau-x* is the same as Kendall's *tau-a* and *tau-b* when measuring correlation between two rankings which contain no ties but it is different from both when ties are present.

Kendall's Tau (Ties Not Permitted)

23. Let us assume for the moment that ties are not permitted. Given two different rankings of n objects - call them R_k and R_l – one can migrate from R_k to R_l over a number of steps by progressively reversing (or "flipping") adjacent pairs of objects. For example, consider the following two rankings of five objects: $R_1 = < 1\ 5\ 3\ 2\ 4 >$ and $R_2 = <5\ 3\ 1\ 4\ 2 >$. The following sequence accomplishes the transformation using 3 flips.

R_1 = <1 5 3 2 4>
<5 1 3 2 4> (flip 1 & 5)
<5 3 1 2 4> (flip 1 & 3)
<5 3 1 4 2> = R_2 (flip 2 & 4)

24. For any pair of rankings $\{R_k, R_l\}$ there is a minimum number of flips required, which we shall label $S(R_k, R_l)$. Similar rankings will require fewer flips to transform one into the other; dissimilar rankings will require more. S is a logical and natural metric to use in the space of all rankings of n objects. Indeed, it is difficult to imagine any other metric that could be more natural.

25. Converting S to a correlation coefficient is straightforward. Let us label it $\Gamma_S(R_k, R_l)$. A value of $S = 0$ indicates identical rankings and is mapped to a correlation value of 1.0. The maximum number of flips possible is $n(n\text{-}1)/2$, corresponding to a complete reversal of ranks, which is mapped to a correlation value of -1.0. Linear interpolation between these extremes leads to the following correlation coefficient.

$$\Gamma_S(R_k, R_l) = 1 - \frac{2\,S(R_k, R_l)}{n(n-1)/2} \qquad (2)$$

26. For the example above, R_1 and R_2 are three flips apart out of a maximum of 10 giving a correlation coefficient of 0.4.

$$\Gamma_S(R_1, R_2) = 1 - \frac{2(3)}{10} = 0.4$$

27. Equation (2) is functionally the same as the formula for Kendall's *tau* despite the fact that the variable used in Kendall's *tau* is not the same as $S(R_k, R_l)$. Kendall's variable is the net score derived from comparing all possible pairs of objects in the two rankings assigning +1 for concordant pairs (where both rankings have the two objects in the same relative order) and –1 for discordant pairs (where they disagree on the relative order). Kendall gave a proof that the variables are equivalent. The proof involves matching any discordant pair of objects with a possible reversal of adjacent objects in a ranking. The heart of the proof requires showing that given any two rankings, there is always at least

one pair of discordant objects which are adjacent.

28. Kendall gave another derivation of *tau* using a more general technique involving ranking matrices. The ranking matrix associated with a ranking of n objects is given by the n by n matrix $\{a_{ij}\}$, where $1 \le i,j \le n$. The elements of the ranking matrix are scores derived from the relative order of the objects in the ranking. Different correlation coefficients result from different ways of assigning these scores. For Kendall's *tau*, the scores in the ranking matrix are defined as follows.

$a_{ij} = 1$ if object i is ranked ahead of object j
$a_{ij} = -1$ if object i is ranked behind object j
$a_{ij} = 0$ if i equals j

29. It can be seen that every ranking corresponds to a unique ranking matrix. Given a ranking matrix, we can resurrect the original ranking using the definition and working backwards. Note, however, that not every matrix with a zero diagonal and diagonally opposite 1's and –1's is necessarily a valid ranking matrix.

30. As an example, the Kendall's *tau* ranking matrices associated with R_1 and R_2 above are given below.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

31. Note that $a_{ij} = -a_{ji}$ by definition for Kendall's *tau*. Hence every value of 1 in the *tau* ranking matrix has a -1 diagonally opposite, and vice versa. The calculation of the rank correlation between two rankings can be completed quickly and neatly by forming the dot product of the two ranking matrices and dividing by the maximum value, which is $n(n-1)$. If ranking R_k is represented by the Kendall's *tau* ranking matrix $\{a^k_{ij}\}$ and R_l by $\{a^l_{ij}\}$ we can write the formula for *tau* as follows.

$$\tau(R_k, R_l) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{k} a_{ij}^{l}}{n(n-1)} \qquad (3)$$

32. Continuing with our example, evaluating the element by element product of matrices M_1 and M_2 above once again gives a value of 0.4 for the Kendall's *tau* rank correlation between rankings R_1 and R_2

$$\tau(R_1, R_2) = 8/20 = 0.4$$

Extending Kendall's Tau When Ties Are Permitted

33. Let us now generalise and permit rankings to contain ties. Does the flip analogy carry over into this expanded domain? It does, but it requires the introduction of the concept of the "half-flip". Two adjacent objects in a ranking which are untied with any others can be half-flipped into a tie with each other. They can be half-flipped again to complete the reversal. Two half-flips equals a whole flip.

34. Suppose a number of objects are tied in a ranking. To separate off one of the objects above or below the others requires "untying" it by half-flipping it with each of the others with which it is tied. Within a large group of p tied objects, any subset of size q can be separated above the rest by half-flipping each of the q objects simultaneously with each of the $(p\text{-}q)$ remaining objects. This requires $q(p\text{-}q)$ half-flips. Note that this operation creates a new rank position between the original tied group and any object(s) that were ranked just above it.

35. To illustrate, consider these rankings of 7 objects: $R_1 = <\ 1\ {-2}\ {-3}\ {-4}\ {-5}\ 6\ 7\ >$ and $R_2 = <\ 4\ 1\ {-3}\ 2\ {-5}\ {-6}\ {-7}>$. One can migrate from R_1 to R_2 as follows:

$R_1 = <\ 1\ {-2}\ {-3}\ {-4}\ {-5}\ 6\ 7\ >$	
$<\ 1\ {-3}\ {-4}\ 2\ {-5}\ 6\ {-7}\ >$	(Split tied group with 3x2 = 6 half-flips)
$<\ 4\ \ 1\ {-3}\ 2\ {-5}\ 6\ {-7}\ >$	(Split off object 4 with 2 half-flips)
$<\ 4\ 1\ {-3}\ 2\ {-5}\ {-6}\ {-7}\ > = R_2$	(Tie last two pairs with 2x2 = 4 half-flips)

The migration from R_1 to R_2 required a total of 12 half-flips to accomplish.

36. The definitions of S and Γ_S carry over. $S(R_k, R_l)$ is the minimum number of flips - where two half-flips equals one whole flip - required to migrate from ranking R_k to R_l (or vice versa). S is not a particularly easy function to calculate, so it would be convenient to have a matrix-oriented calculation method. Kendall's *tau* performs nicely in the no-ties environment, but can it be generalised to accommodate ties? We will demonstrate that it can by introducing the rank correlation statistic ***tau-x*** – where the "*x*" stands for "extended" - which we shall shorten to τ_x.

37. In order to derive a suitable score for tied elements in the ranking matrix, we will take the approach of simply solving for the matrix entry value that satisfies the appropriate conditions. For non-tied elements, the ranking matrix elements are defined as for Kendall's *tau*.

$a_{ij} = 1$ if object i is ranked ahead of object j
$a_{ij} = -1$ if object i is ranked behind object j, and
$a_{ij} = 0$ if i equals j

For tied elements we let the ranking matrix score be the unknown value c.

$a_{ij} = c$ if object i is tied with object j

38. Consider the smallest non-trivial ranking space - the space of all rankings of two objects. There are three possible rankings: $R_1 = < 1\ 2 >$, $R_2 = < 1\ -2 >$, and $R_3 = < 2\ 1 >$. These are given in ranking matrix form below.

$$M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

39. It is axiomatic that any given ranking requires zero flips to be transformed into itself. Hence, our extended *tau* computation must have the property that every ranking has correlation 1.0 with itself. In particular, if we take the dot product of ranking matrix M_2 with itself we obtain the following.

$$\begin{aligned}\tau_X(R_2,R_2) &= [0*0 + (c)(c) + (c)(c) + 0*0] / [2(2-1)] \\ &= 2\,c^2 / 2 \\ &= c^2 = 1.0\end{aligned}$$

Therefore $c = +1$ or $c = -1$.

40. We will adopt the $c = +1$ convention. Note that Kendall's approach in handling ties was different. Instead of using +1 or −1, he defined *tau-a* and *tau-b* by assigning a value of 0 to a_{ij} when object i was tied with object j. This permitted matrix symmetry to be maintained (a_{ij} always equals $-a_{ji}$), but led to inconsistencies and problems. Any ranking with ties could not achieve a correlation of 1.0 with itself unless the denominator was reduced as it is in *tau-b*. This led to the untenable situation where the correlation of the unique all-ties ranking with itself was undefined. Indeed, for Kendall's *tau-b* the all-ties ranking has zero correlation with every other ranking. This makes no sense, as surely a ranking which has all objects tied except one should be much closer to the all-ties ranking than a ranking that has no ties whatsoever.

41. Summarising, the *tau-x* ranking matrix for the generalised case will become the following.

$a_{ij} = 1$ if object i is ranked ahead of **or tied with** object j
$a_{ij} = -1$ if object i is ranked behind object j, and
$a_{ij} = 0$ if i equals j

42. The final step is to prove that Γ_S and τ_X, calculated with the generalised formulation of Equation (2), are equivalent. As with the no-ties case, the proof involves using the "discordant pair" concept, defined as any pair of objects for which two given rankings disagree on their relative placement. From the perspective of one of the rankings, any discordant pair can be rectified by flipping or half-flipping the two objects into agreement (along with all other objects tied with either object which must also form discordant pairs), as long as the two objects are adjacent or tied in that ranking. The heart of the proof, presented in complete detail in Annex A, requires showing that there will always be at least one pair of discordant objects which are adjacent.

III. ISSUES WITH KENDALL'S TAU AND SPEARMAN'S RHO

43. At this point we will discuss our experience with Kendall's *tau* and Spearman's *rho* as rank correlation measures. Insurmountable problems exist with both in the context of the weighted *m*-rankings problem. These will be detailed using simple examples. The rejection of Kendall's *tau* in particular was the motivating force for developing *tau-x* as a new rank correlation coefficient ideally suited for the weighted *m*-rankings problem.

Problems With Kendall's Tau

44. As mentioned earlier, Kendall's *tau* as a measure of agreement between two rankings is based on a comparison of the relative order of all pairs of objects. It therefore uses the ordinal data of a ranking in a theoretically sound way. Kendall's *tau-a* and *tau-b* are very similar to *tau-x*, being identical when there are no ties in either ranking. *Tau-a* and *tau-b* differ in the way they are normalized but both assign a zero score to tied objects. *Tau-a* uses the same normalization as in the no-ties case and consequently has the undesirable property that any ranking containing ties has correlation of less than 1 with itself. *Tau-b* is normalized in a more complex manner to avoid this problem. Unfortunately the normalization of *tau-b* creates more subtle problems. The reader is referred to Reference 1 for complete details of the calculation of *tau-a* and *tau-b*.

45. The following simple example illustrates the anomalous behaviour we have observed when using Kendall's *tau-b* in the *m*-rankings problem. First we will consider three rankings of two objects and then we will add a third object to show how *tau-b* gives an illogical result.

Ranking 1 :	< 1 2 >	Weight = 1
Ranking 2 :	< 1 2 >	Weight = 1
Ranking 3 :	< 2 1 >	Weight = 1

46. It is evident for this simple case that object 1 should be ranked ahead of object 2 since two out of the three rankings have this order. The solution using *tau-b* as the

measure of agreement can be found by finding the ranking which gives maximum average *tau-b* rank correlation with the three input rankings. We will illustrate this by doing an exhaustive search. In this case there are only three possible solution rankings.

Possible Solutions	Average Value of *Tau-b*
< 1 2 >	0.33
< 1 -2 >	0.00
< 2 1 >	-0.33

47. The maximum average *tau-b* solution is the ranking < 1 2 > which gives an average correlation of 0.33 with the three input rankings, in agreement with the discussion above. We now add another object, designated as object 3.

Ranking 1a :	< 1 2 3 >	Weight = 1
Ranking 2a :	< 1 2 3 >	Weight = 1
Ranking 3a :	< 2 1 3 >	Weight = 1

48. Note that object 3 is ranked last in all three of these rankings and objects 1 and 2 have the same ranking as before. Since this is the case, it is clear that object 3 could be removed from each of the rankings resulting in a set of three rankings of two objects identical to the case above in which the solution ranking had object 1 ahead of object 2. Therefore the solution to the expanded problem should logically be the same as before with object 3 added at the end, i.e. the ranking < 1 2 3 >. However, this is not what happens when we find the maximum average *tau-b* solution for this case.

49. To find the ranking which gives the maximum average agreement as measured by Kendall's *tau-b* we again calculate the average value of *tau-b* with all possible solution rankings. There are 13 possible rankings of three objects. The average value of *tau-b* for each is presented in Table I. Also given in Table I is the average value of *tau-x* for each possible ranking for comparison.

50. As the table indicates, the maximum average value of *tau-b* is given by the solution ranking < 1 -2 3 > in which object 3 is last as expected, but objects 1 and 2 are now illogically tied for first. This behavior is clearly anomalous and

demonstrates the inappropriateness of Kendall's *tau-b* as a metric in the space of all possible solution rankings.

Table I

Average Value of Tau-B For All 13 Possible Solutions

Possible Solutions	Average Value of *Tau-b*	Average Value of *Tau-x*
< 1 2 3 >	.78	.78
< 1 –2 3 >	.82	.67
< 2 1 3 >	.56	.56
< 1 2 –3 >	.54	.44
< 2 1 –3 >	.27	.22
< 1 3 2 >	.11	.11
< 1 –2 –3 >	.00	.00
< 2 3 1 >	-.11	-.11
< 1 –3 2 >	-.27	-.22
< 2 –3 1 >	-.54	-.44
< 3 1 2 >	-.56	-.56
< 3 1 –2 >	-.82	-.67
< 3 2 1 >	-.78	-.78

51. Note that using the new rank correlation coefficient *tau-x*, the solution ranking for the above case is < 1 2 3 > which is what one would expect.

Problems With Spearman's Rho

52. Spearman's *rho* is another popular rank correlation coefficient named after C. Spearman who introduced it. (Chapter 2 of Reference 1 gives a mathematical derivation of *rho* using ranking matrix scores based on rank separations.) Spearman's *rho* is defined in terms of differences between the rank values assigned to pairs of objects. For this

reason when Spearman's *rho* is calculated, the rank value of adjacent objects must differ by exactly one unit and tied objects must be given the average of the corresponding rank values. Even with these conventions, the use of rank values as if they were mathematical variates leads to anomalous behaviour.

53. Kendall shows in Reference 1 that when the weights are equal and no ties are present in any of the rankings, maximizing the average Spearman's *rho* rank correlation with *m* given rankings is equivalent to selecting that ranking which is derived using rank-sum as a criterion. For each object, the sum of the *m* ranks associated with that object is computed. The object with smallest rank-sum is placed first, and so on. Note that when ties are present in the rankings the rank-sum method does not give the maximum Spearman's *rho* solution.

54. The problem with Spearman's *rho* in the weighted *m*-rankings problem context is that this coefficient gives illogical results due to the technical flaw known as sensitivity to irrelevant alternatives. (An irrelevant alternative is one that is asymmetrically dominated. This means that the object is less preferred in every ranking to another object but not by every other object.) A simple case will be illustrated but the reader should be aware that the underlying flaw with Spearman's *rho* can lead to very unsatisfactory and illogical results when used as a measure of agreement between rankings in the weighted *m*-rankings problem.

55. Consider the following case of three equally weighted rankings of two objects.

Ranking 1 :	< 1 2 >	weight = 1
Ranking 2 :	< 1 2 >	weight = 1
Ranking 3 :	< 2 1 >	weight = 1

Because two of the three rankings show object 1 preferred over object 2, it is evident that the most reasonable solution is the ranking < 1 2 > and this can be shown to be the case using Spearman's *rho* as the measure of agreement. However, suppose ranker 3 wishes to change the outcome. He can do so if he can add two extra objects to the ranking exercise, even if these objects are irrelevant. Suppose two token objects are added and assume that rankers 1 and 2 both place the two token objects in a tie for last.

Ranker 3 places the two extra objects behind object 2 and ahead of object 1, resulting in the three rankings shown below.

Ranking 1a :	< 1 2 3 -4 >	weight = 1
Ranking 2a :	< 1 2 3 -4 >	weight = 1
Ranking 3a :	< 2 3 -4 1 >	weight = 1

56. The maximum agreement solution to this 3-rankings problem using Spearman's *rho* as the metric is the ranking < 2 1 3 -4 >. Note that object 2 has moved to first place. By adding two token objects Ranker 3 has changed the outcome by taking advantage of the behaviour of Spearman's *rho* when used as a measure of agreement in the *m*-rankings problem. This anomaly appears to occur because Spearman's *rho* uses the ranks as if they were variate values instead of purely order values. Because of this sensitivity to irrelevant alternatives, Spearman's *rho* is not suitable as a rank correlation coefficient in the weighted *m*-rankings problem. Neither Kendall's *tau* nor the new rank correlation coefficient *tau-x* is sensitive to irrelevant alternatives, both being based on rank order only and not on the value of the ranks.

57. *Tau-x* does not display the anomalous behaviour exhibited by Kendall's *tau* nor is it sensitive to irrelevant alternatives as is Spearman's *rho*. We therefore conclude that *tau-x* is a more appropriate rank correlation coefficient to use as a measure of agreement between rankings in the weighted *m*-rankings problem.

IV. COMPUTER IMPLEMENTATION OF OUR SOLUTION CONCEPT

Enumerating All Rankings Of N Objects

58. Our initial attempts to apply *tau-x* as a measure of agreement in the weighted *m*-rankings problem centered on the use of brute force enumeration of all possible solution rankings using fast computers. We quickly ran into a wall when the number of objects to be ranked was in the range of 10 or more. To see this, it is of interest to calculate the size of the solution space in which we are working. With no ties allowed, the total number of rankings of *n* objects is given by *n* factorial. When ties are allowed the number of rankings can be found by means of the following difference equation. (Reference 4)

$$Q_n = \sum_{i=1}^{n} i \left(q_{n-1}^{i-1} + q_{n-1}^{i} \right)$$

Q_n is the total number of rankings of *n* objects and q_k^j is the number of rankings of k objects which have exactly j indifference sets.

59. An indifference set is defined as either a single object or a tied set of objects. (Reference 2) For $n = 2$ objects we have a total of 3 possible rankings, one of which has one indifference set and two of which have two indifference sets. Applying the difference equation we find that the number of rankings for $n = 3$ objects consists of 1 ranking with one indifference set, six rankings with two indifference sets, and six rankings with three indifference sets for a total of 13. Proceeding in a similar way we find that there are 75 total rankings for $n = 4$, and 541 total rankings for $n = 5$. More values are given in Table II below.

TABLE II
NUMBER OF RANKINGS OF N OBJECTS WITH TIES

Number of Objects (n)	Total Number of Rankings With Ties (Q_n)
2	3
3	13
4	75
5	541
6	4683
7	47293
8	545835
9	7087261
10	102247563
11	1622632573

60. It is clear that the total number of rankings grows very large very quickly. An asymptotic formula for the total number of rankings of n objects is given below (Reference 2). This formula is remarkably accurate for even moderate values of n. The limit as n approaches infinity is

$$Q_n \rightarrow \frac{C^{n+1}\,(n!)}{2}$$

$$\text{where } C = \frac{1}{\ln 2} = 1.44269504088963...$$

61. As can be seen from the above formula, the solution space grows as the product of an exponential and a factorial. It is clearly hopeless to rely on exhaustive enumeration as

a solution technique. Fortunately we have been able to develop a branch and bound algorithm which handles most problems where the number of objects being ranked is as large as n=20 to n=30. For larger values of n, we have developed a heuristic method which works by repeatedly finding the best ordering of subsets of options.

Branch and Bound Solution

62. In order to develop our solution method, we must re-visit the ranking matrix idea developed earlier. First let us re-write the weighted m-rankings problem using ranking matrices to represent the input rankings.

Label	Weight	Ranking Matrix
R_1	w_1	$\{a_{ij}^1\}$
R_2	w_2	$\{a_{ij}^2\}$
.	.	.
.	.	.
.	.	.
R_m	w_m	$\{a_{ij}^m\}$

63. Let the ranking matrix for a solution ranking be given by $\{s_{ij}\}$. The value of the *tau-x* rank correlation between the solution ranking and the kth input ranking may be written as follows.

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} a_{ij}^k}{n(n-1)}$$

64. The function we want to maximize is given by the weighted sum of the *tau-x* rank correlation of each of the m input rankings with the solution ranking. We may write this

function as follows.

$$\frac{w_1 \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} a_{ij}^{1} \;+\; w_2 \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} a_{ij}^{2} \;+\ldots+\; w_m \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij} a_{ij}^{m}}{n(n-1)\sum_{k=1}^{m} w_k}$$

65. This expression can be further simplified. The normalizing factor in the denominator can be ignored since it plays no role in the maximizing process.

$$\text{Maximize}: \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}\left(\sum_{k=1}^{m} w_k a_{ij}^{k}\right) \qquad (4)$$

66. Inspection of expression (4) reveals that the maximization problem can be reduced to the problem of finding a ranking matrix $\{s_{ij}\}$ which when multiplied term by term with the matrix $\left\{\sum_{k=1}^{m} w_k a_{ij}^{k}\right\}$ gives a maximum. We shall call this latter matrix the combined input matrix, or simply the CI matrix. We note that if the CI matrix happens to correspond to a valid ranking of *n* objects in terms of the pattern of positive and negative values of its elements, then this ranking will be the solution because it obviously maximizes the dot product summation. In general, however, the CI matrix will rarely correspond to any valid ranking of *n* objects. The problem then becomes one of searching the ranking space for a ranking whose ranking matrix, when multiplied element by element with the CI matrix, gives a maximum value.

67. One additional point about expression (4) can be made. From inspection it appears at first that since the CI matrix part of expression (4) is a constant, it may be possible to maximize the expression by considering each pair of solution matrix elements in isolation. The problem with this approach is that it ignores the constraint that the solution matrix must correspond to a valid ranking of *n* objects. Note that in a valid ranking matrix diagonally opposite elements cannot both be negative. This property also

applies to the CI matrix.

68. An important point which can be seen from the mathematical description of the weighted m-rankings problem is that neither the addition of weights nor increasing the number m of input rankings affects the complexity of finding a solution. All the information from the input rankings is contained in the n by n CI matrix so that the complexity of finding a solution depends only on the number of objects being ranked.

69. To implement our branch and bound solution we have found it more convenient to approach the problem from a slightly different point of view. We first calculate the maximum possible value of expression (4) by summing the absolute value of all the elements of the CI matrix. Any given candidate solution ranking when applied to expression (4) will result in a value which is less than or equal to this maximum. We define the difference between the maximum value and the actual value to be the penalty associated with a candidate solution ranking. Our task then is to find the ranking which gives minimum penalty.

70. The branch and bound algorithm requires a starting solution. Finding an appropriate starting solution will be discussed below. For now, assume we have an initial solution for which the penalty as defined above is P_0. Our algorithm consists of evaluating all possible rankings of n objects, searching for a penalty lower than P_0. We start by noting that there are three possible orderings of the first two objects in the initial solution: they are either tied or one is ranked ahead of the other. We choose each of these in turn and calculate the penalty P incurred by the choice. Penalty P can be calculated using the appropriate elements of the CI matrix, i.e. those that correspond to the two objects in question. If penalty P is greater than the initial penalty P_0, we can stop evaluating this branch because clearly all rankings on this particular branch will have a total penalty which exceeds P_0. If P is less than or equal to P_0, we continue by considering the next highest ranked object in the initial solution. In the case where the first two objects are untied, this object can be placed behind the first two, tied with the second, between the two, tied with the first, or ahead of both. Again for each of these possibilities we calculate the additional penalty incurred in the placement of this object with respect to the others by using the appropriate elements of the CI matrix. Having found this additional penalty we add it to the previous penalty for the branch. We

compare the updated penalty for the current branch to P_0 and take the appropriate action as before, either abandoning the rest of the branch or going on.

71. We proceed in this way, going down each branch until we either cut off the rest of the branch or arrive at a solution ranking which is tied with or better than the initial solution. If we find a lower value of P_0 we update our minimum value and proceed with this updated value until all branches have been evaluated. In this way we can guarantee to eventually find all rankings which give the minimum penalty for the particular problem.

Practical Considerations

72. **Run Time**. The branch and bound algorithm described above has been implemented on a computer and has been found to work well for problems with number of objects n up to 20 to 30, depending on how well the input rankings agree with each other. With strong consensus our algorithm can evaluate a 30 object problem in less than an hour on a Pentium PC. With weaker consensus the algorithm may take weeks to run a 30 object problem. Problems with 20 objects and modest consensus can be run in under an hour. For problems with 15 or fewer objects the algorithm takes under an hour even for random rankings.

73. **Partitioning**. In some cases it is possible to reduce a large problem to a smaller one by partitioning the CI matrix. For example, if a row of the matrix has every element except the diagonal strictly less than zero, the object represented by the row can be placed last in the solution ranking and there will be zero penalty associated with this assignment. The object can be removed from the problem, reducing it from size n to size n-1. Sets of more than one object can sometimes be removed from the problem in this fashion and solved separately. Again this involves a zero penalty partition of the combined input matrix.

74. **Initial Solution**. The run time of most problems is sensitive to the quality of the initial solution. The lower the initial penalty, the faster the algorithm can progress through the ranking space. Problems that run in under an hour with a good initial

solution may take days to run with a poor initial solution. Several methods for finding a good initial solution were tried. Simply choosing one of the *m* input rankings as the initial solution gave very poor results. Choosing an initial solution by treating the ranks as variates and ordering the objects according to rank-sum gave inconsistent results. Heuristic methods based on processing the CI matrix provided generally satisfactory initial solutions. However the best method we have found is to apply the solution heuristic described below which was developed for large problems. This heuristic consistently gives excellent initial solutions for all problem sizes.

75. **Multiple Solutions.** In most problems the number of solutions is small if there is a reasonable level of consensus in the input rankings. If there is little or no consensus or if there is frequent occurrence of objects missing in the input rankings, the number of solutions can be large. This should be a relatively rare situation but could be problematical in some cases. The user should be aware of the possibility of numerous solutions and the necessity of handling the problem.

A Heuristic Method To Apply With Large N (Greater Than 20 or 30)

76. As mentioned above, we have developed a heuristic method for obtaining a solution which works by repeatedly finding the best ranking within subsets of the objects. We have named it the Slice program. The Slice program can be used to find a very good starting point for the branch and bound program, sometimes producing the best ranking by itself.

77. The Slice method takes advantage of the fact that the branch and bound algorithm runs in negligible time when the number of objects is small. The method consists of applying the branch and bound algorithm to the first 11 objects in an initial solution as if there were only 11 objects being ranked in total. Having placed these objects in the best order using the branch and bound algorithm we process the next set of 11 objects which is the same as the previous 11 except that the object now ranked first is omitted and the next object from the initial ranking is added. The process is repeated, deleting and adding one object at a time until we reach the last set of 11 objects. Then we go back to the beginning and repeat this iterative "slicing" process on the ranking which emerged from

the initial pass of the program. The entire process is repeated until there is a pass through the objects in which no changes occur.

78. The Slice program runs relatively quickly and for some problems will find the best overall solution by itself. The Slice algorithm is designed to search for all the solutions for cases where there are more than one. This has worked well in practice but the only way to verify that we have all the solutions is to run the branch and bound algorithm. This is also necessary if one wishes to guarantee that the ranking produced by Slice is in fact the optimal solution. For cases where this verification is not practical we have modified Slice to run several times with random initial ordering of the objects. The choice of initial ranking may limit how close the algorithm can get to the optimal solution. Multiple executions of the Slice algorithm with different, randomly selected, initial rankings may help home in on better (ie. lower penalty) solutions.

Dealing With Partial Rankings

79. It sometimes happens that for one reason or another, some of the input rankings may not be complete. This could happen if the ranking information for certain objects is not available or impossible to obtain. In other cases a human ranker may be unable or unwilling to rank one or more objects.

80. If missing rankings occur only occasionally, our solution methods are able to easily accommodate the problem. The CI matrix consists of the weighted sum of the comparison scores for each input ranking and its elements represent all of the available ranking information. If an input ranking does not contain a given object, the ranking makes no contribution to the combined input matrix elements which involve that object. The CI matrix will receive no contribution from this ranking with respect to the missing object and therefore only those rankings in which an object is ranked will determine the final placement of that object in the solution ranking.

81. It should be noted that if there is massive missing data the ranking of a given object in the solution will be determined only by those input rankings which contain the object, making it possible in extreme cases for a single ranking to determine the overall

winner. Great care is recommended in dealing with such situations.

82. On a more technical note, it is of interest to observe that off-diagonal zeros in a ranking matrix represent missing data in our methods whereas they represent tied objects in Kendall's original work. Also we note that an incomplete ranking cannot logically be in full agreement with any other ranking including itself because of the missing information. Our new *tau-x* rank correlation coefficient gives results in agreement with this observation.

V. A DETAILED EXAMPLE

83. To illustrate the solution techniques discussed above we will analyze an example in depth. The example is an abridged version of a real case of the type described in the introduction under Strategic Priorities for National Defence. The example has three sets of seven rankings representing the expert opinions of seven equally weighted individuals. The three sets correspond to three separate ranking criteria, each of which has a weight based on its overall importance. The objects being ranked were 15 departmental initiatives which were competing for limited funds.

84. From the listing of the input rankings in Table III, it can be seen that the example contains some incomplete rankings. For example ranking R1 does not have objects 14 and 15 and rankings R16, R18 and R19 lack objects 1 and 2.

85. It is of interest to note that the structure of the example is such that one could consider the problem as consisting of three weighted sub-problems, each of which consists of seven equally weighted rankings of 15 objects. In fact this is what happened in the real case. We solved each sub-problem and then used the seven solution rankings as input rankings with appropriate weights to determine the final solution. However, it was quickly pointed out that this technique is flawed and not appropriate because the intermediate solutions mask the disagreement present in the original rankings so that the final solution may be radically different from the solution obtained by using the 21 original weighted rankings as input.

86. To illustrate this, consider a simple example with three sets of two rankings, all equally weighted for simplicity.

Set 1 Rankings	Set 2 Rankings	Set 3 Rankings
< 1 2 >	< 1 2 >	< 2 1 >
< 1 2 >	< 1 2 >	< 2 1 >
< 2 1 >	< 2 1 >	< 2 1 >

Table III

Input Rankings For Detailed Example

NO.	INPUT RANKING
	Input Rankings for First Criterion Weight = 4
R1.	< 6 1 -7 -9 10 3 8 11 5 -12 2 -4 -13 >
R2.	< 6 10 9 3 4 -8 7 1 -5 -11 2 12 13 14 15 >
R3.	< 6 10 3 7 8 11 5 14 15 12 1 -4 -13 2 -9 >
R4.	< 6 9 -11 10 3 14 12 7 4 5 2 1 8 13 15 >
R5.	< 10 6 7 1 11 -13 4 2 3 9 12 14 -15 8 5 >
R6.	< 6 9 8 -10 11 4 1 5 7 15 2 12 14 13 3 >
R7.	< 1 -6 -10 7 -12 9 3 4 -11 -14 -15 2 -13 8 >
	Input Rankings for Second Criterion Weight = 5
R8.	< 4 -10 1 -7 6 -9 -13 5 -14 3 12 8 11 -15 2 >
R9.	< 4 -9 5 1 14 11 8 3 6 2 -13 10 12 7 15 >
R10.	< 4 2 -5 8 15 7 11 -14 1 -12 -13 10 9 6 >
R11.	< 2 -11 -12 -14 -15 6 -13 3 -4 9 8 -10 1 -5 -7 >
R12.	< 4 14 10 2 5 3 1 13 12 7 15 8 11 6 9 >
R13.	< 4 2 5 1 15 7 13 14 3 -12 8 11 6 9 10 >
R14	< 12 1 3 -4 2 11 -13 -15 9 14 6 8 7 -10 5 >
	Input Rankings for Third Criterion Weight = 7
R15.	< 5 4 9 2 -7 14 8 -11 3 1 15 12 6 10 13 >
R16.	< 11 9 -14 15 12 3 4 13 8 6 7 10 5 >
R17.	< 12 11 2 1 3 9 8 10 13 -14 6 4 -15 5 7 >
R18.	< 4 -5 10 -12 3 8 -11 6 -7 -9 13 14 15 >
R19.	< 12 5 -13 14 3 8 15 4 9 -10 11 6 7 >
R20.	< 4 -5 -8 11 6 14 7 1 -2 -15 10 3 13 9 -12 >
R21.	< 10 8 5 -11 6 -14 9 4 -13 -15 3 -12 2 1 >

87. For sets 1 and 2 the solution ranking is clearly < 1 2 > and for Set 3 the solution ranking is < 2 1 >. If we proceed to combine these three solution rankings we get the final solution < 1 2 > . However, if we use all 9 input rankings in one combined problem we get the opposite solution < 2 1 >, illustrating that if we use the intermediate solution method we get an incorrect answer because the intermediate solutions have masked the lack of unanimity in Sets 1 and 2.

88. The first step in finding a solution to the example problem is to create the ranking matrix for each of the 21 input rankings. Then these are combined as explained earlier to create the combined input matrix presented in Table IV. Each ranking has a weight in accordance with the weights of the three criterion sets.

89. Inspection of the CI matrix shows that we can remove object 4 from the problem in accordance with our earlier remarks on partitioning. Note that every element (other than the diagonal element) in row 4 of the combined input matrix is positive and every element (other than the diagonal element) in column 4 is negative. This means that object 4 is ranked first overall in the solution. Using similar reasoning it can also be seen that objects 13 and 15 can be removed and placed in a tie for last place.

90. After removing the three objects we are left with a 12 object problem which cannot be further partitioned. We must therefore evaluate the 28 billion possible rankings of 12 objects to determine which give the highest weighted average *tau-x* correlation with the input rankings. The branch and bound algorithm described earlier was used and was able to find three solutions in a few seconds. The three solutions are given below.

Solution 1.	<	4	10	5-11	1	-2-14	3-12	9	8	6	7	13-15	>
Solution 2.	<	4	10	5-11	1	-2	9	14	3-12	8	6	7	13-15	>
Solution 3.	<	4	10	5-11	2-14		1	3-12	9	8	6	7	13-15	>

91. Each of the solutions gives a weighted average *tau-x* correlation of 0.16556 with the input rankings and no other ranking does as well. The three solution rankings have the same four objects ranked first and the same five objects ranked last. The placement of the middle six objects differs slightly in the three solution rankings.

Table IV

Combined Input Matrix for the Example Problem

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	14	-35	-11	5	14	5	19	7	-9	27	67	-1	45
2	13	0	2	-59	-15	-13	4	11	3	1	-27	11	51	5	39
3	-14	-2	0	-5	-15	3	20	47	5	-11	-7	15	33	-25	33
4	43	67	25	0	50	18	65	46	30	28	20	18	66	26	62
5	29	25	15	-22	0	2	27	6	0	-22	20	30	42	30	48
6	3	13	-3	-18	-2	0	51	-26	26	18	-46	-8	30	-8	16
7	14	10	-20	-65	-17	-37	0	-21	-5	-25	-11	1	23	-3	19
8	-5	-11	-47	-24	8	26	21	0	-18	16	30	-18	18	-18	38
9	-11	5	-5	-20	0	-2	27	18	0	8	-24	4	18	18	22
10	1	-1	11	-18	22	-10	35	2	6	0	-2	16	34	-8	12
11	17	37	7	-12	2	46	11	-2	32	2	0	22	68	40	64
12	-17	-1	9	-18	-22	8	7	18	10	-2	-12	0	36	-14	16
13	-49	-25	-33	-36	-28	-10	-23	-18	-8	-34	-50	-26	0	-20	12
14	1	5	25	-18	-20	22	3	18	-4	8	-12	24	34	0	70
15	-31	-15	-33	-26	-48	-16	-19	-38	-22	-12	-26	-6	12	-44	0

Note: Each row of the matrix corresponds to the object listed at the beginning of the row and the same for the columns. The matrix element associated with each row and column is the weighted sum of the *tau-x* comparison scores for the corresponding objects. All of the diagonal elements are zero by definition.

Discussion

92. As can be seen by the above example, we have been successful in applying the solution concept of this paper to real problems in the Canadian Department of National Defence. The methodology has been found to be robust, logical and intuitively appealing. We have extended the utility of the methodology to handle problems with unlimited numbers of weighted rankings while also allowing for incomplete rankings.

The branch and bound algorithm has been used successfully on real cases with 31 ranked objects. We have also applied heuristic methods to handle test problems with hundreds of ranked objects.

Other Applications

93. Although we developed *tau-x* as a measure of agreement specifically for application to the weighted *m*-rankings problem, it is also usable as a general rank correlation coefficient. *Tau-x* has all the compelling intuitive properties of Kendall's *tau* but handles ties in a more logical way and consequently does not suffer from the same difficulties as *tau* when ties are considered. In particular, *tau-x* may be profitably applied to the problem of testing for independence in contingency tables where massive ties are a feature of the problem. (See Annex B.) *Tau-x* may also be used to test for statistical independence between sets of two or more rankings, especially where the rankings may be incomplete.

VI. SUMMARY AND CONCLUSIONS

94. The weighted *m*-rankings problem occurs frequently in the Department of National Defence. Examples include committees such as the Army Equipment Board; multi-criteria decision problems such as transport aircraft color selection; and complex mixes of the two such as a recent exercise to determine strategic priorities. In each case it is required to provide a solution ranking of *n* objects based on *m* weighted input rankings. The input rankings may contain unlimited ties and may also be incomplete.

95. The basic concept employed in this report is to find a ranking which agrees most closely with the set of input rankings, where agreement is measured by a rank correlation coefficient. The solution space is the set of all rankings of *n* objects where ties are allowed. The solution will be a ranking which gives the maximum weighted average correlation with the input rankings.

96. The two best known rank correlation coefficients were tried and ultimately rejected as measures of agreement for the weighted *m*-rankings problem. Kendall's *tau* was found to exhibit unacceptable properties because of the way it handles ties. Spearman's *rho* suffers from sensitivity to irrelevant alternatives. A new rank correlation coefficient based on Kendall's *tau* was developed and found to have all the compelling and logically sound properties of *tau* but handled ties without the logical flaws. We have named the new rank correlation coefficient *tau-x* because it is an extension of Kendall's *tau*. We have given a formal development of *tau-x* based on the concept of interchanges or flips, showing how it arises naturally if the concept of the half-flip is introduced to create or break ties. *Tau-x* has been found to be an ideal measure of agreement in the weighted *m*-rankings problem with the ability to handle ties, weights and incomplete rankings in a sound and logical way.

97. Our solution concept for the weighted *m*-rankings problem requires searching the space of all possible rankings of *n* objects with ties allowed. This space is huge, growing as the product of an exponential and a factorial. We have developed a branch and bound algorithm which has been applied successfully in problems with moderate consensus involving up to 31 ranked objects. For larger problems we have developed a heuristic

algorithm which is capable of handling problems with hundreds of ranked objects.

98. The methodology and techniques we have developed for the weighted m-rankings problem are currently being applied successfully in the Department of National Defence. Our methods provide a mathematically sound and practical approach to the variety of ranking problems which occur so frequently in high level decision support.

REFERENCES

1. Rank Correlation Methods, Fifth Edition, Maurice Kendall and Jean Dickenson Gibbons, 1990 (original edition 1948), Oxford University Press.

2. Preferential Arrangements, O.A. Gross, The RAND Corporation, American Mathematical Monthly, January 1962.

3. The Advanced Theory of Statistics Volume 2, Fourth Edition, Kendall and Stuart, MacMillan Publishing Co., 1979.

4. Maximum Rank Correlation as a Solution Concept in the M Rankings Problem With Application to Multi Criteria Decision Analysis, E. J. Emond, DOR(CAM) Research Note RN 9705, September 1997.

ANNEX A
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PROOF THAT τ_X AND Γ_S ARE EQUIVALENT

The coefficient $\Gamma_S(R_k, R_l)$ is defined as the minimum number of flips or half-flips required to transition from ranking R_k to ranking R_l. The coefficient $\tau_X(R_k, R_l)$ is defined in matrix terms. Any ranking of n objects has an associated n by n ranking matrix defined as $\{ a_{ij}^k \}$, where:

$$\begin{aligned} a_{ij}^k &= 1 \quad \text{if object } i \text{ is ranked before or tied with object } j \\ a_{ij}^k &= -1 \quad \text{if object } i \text{ is ranked behind object } j, \text{ and} \\ a_{ij}^k &= 0 \quad \text{if } i \text{ equals } j \end{aligned} \tag{A1}$$

We may write τ_X as follows.

$$\tau_X(R_k, R_l) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^k a_{ij}^l}{n(n-1)} \tag{A2}$$

Effectively, τ_X uses the multiplicative properties of the integers +1 and –1 to enumerate the mismatches between the ranking matrices associated with R_k and R_l. A "discordant pair" of objects is defined as any pair of objects that do not have the same relative ranking in both rankings, and hence will have different associated matrix entries. For example, ranking R_k may have object i ranked ahead of object j, whereas ranking R_l may have them tied. Hence $a_{ij}^k = 1$ and $a_{ji}^k = -1$ in the ranking matrix for R_k, whereas $a_{ij}^l = a_{ji}^l = 1$ in the ranking matrix for R_l. There is a duality between the single mismatched matrix position and a potential half-flip of the two objects corresponding to that position. What has to be proven is that one can navigate with flips or half-flips (possibly in simultaneous groups) from one ranking to the other such that only half-flips which correct matrix entry mismatches are necessary. If we can prove such a direct path must exist, then we will have shown the equivalence of τ_X and Γ_S. This distills to the

requirement to prove the following theorem.

THEOREM 1 : Given any pair of rankings, R_1 and R_2, of n objects with ties permitted, there must be in either ranking at least one discordant pair of objects which are adjacent or tied.

PROOF : Assume the converse is true – that there exists a situation where none of the discordant pairs are adjacent or tied in one of the rankings, say R_1. From all the discordant pairs, select that pair which R_1 has ranked most closely together, with "closeness" being measured by the number of intervening objects ranked between them. Label this pair (a,b) with a being ranked ahead of b. Select one of these intervening objects and call it x. Let the rank values of the three objects be $\pi(a)$, $\pi(b)$, and $\pi(x)$. R_1 must have a, b, and x ranked in the following order.

$$R_1 : \pi(a) < \pi(x) < \pi(b)$$

Object x cannot be discordant with either a or b. If it were, then (a,x) or (b,x) would be a discordant pair closer together than (a,b), contradicting our selection criterion for (a,b). Since a and b are discordant, ranking R_2 must have the two equally ranked, or have b ahead of a. Each case is considered separately below.

Case 1 : a , b tied in R_2

There are three possiblities for where x can fit in relation to a and b in ranking R_2 .

i) $R_2 : \pi(a) = \pi(b) < \pi(x)$

R_1 has $\pi(x) < \pi(b)$ which makes (b,x) a discordant pair, a contradiction.

ii) $R_2 : \pi(a) = \pi(b) = \pi(x)$

R_1 has $\pi(x) < \pi(b)$ which makes (b,x) a discordant pair, a contradiction.

iii) $R_2: \pi(x) < \pi(a) = \pi(b)$

R_1 has $\pi(a) < \pi(x)$ which makes (a,x) a discordant pair, a contradiction.

Case 2 : b ranked ahead of a in R_2

There are five possiblities for where x can fit in relation to a and b in ranking R_2.

i) $R_2: \pi(x) < \pi(b) < \pi(a)$

R_1 has $\pi(a) < \pi(x)$ which makes (a,x) a discordant pair, a contradiction.

ii) $R_2: \pi(x) = \pi(b) < \pi(a)$

R_1 has $\pi(a) < \pi(x)$ which makes (a,x) a discordant pair, a contradiction.

iii) $R_2: \pi(b) < \pi(x) < \pi(a)$

R_1 has $\pi(a) < \pi(x)$ which makes (a,x) a discordant pair, a contradiction.

iv) $R_2: \pi(b) < \pi(x) = \pi(a)$

R_1 has $\pi(a) < \pi(x)$ which makes (a,x) a discordant pair, a contradiction.

v) $R_2: \pi(b) < \pi(a) < \pi(x)$

R_1 has $\pi(x) < \pi(b)$ which makes (b,x) a discordant pair, a contradiction.

Since all possibilities lead to contradiction our proof is completed.

ANNEX B
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THE USE OF TAU-X IN ORDERED TABLES

In an ordered table the rows and columns are ordered categories and the values in the table are the frequency counts for the corresponding row and column categories. In this type of table we are often interested in the existence of some type of association between the row and column breakdowns. Statistically, we wish to test the hypothesis that the row and column breakdowns are independent. Consider the following example, taken from Reference 3, Section 33.38.

Table of Left and Right Eye Visual Ratings

		Left Eye				
		Highest Grade	Second Grade	Third Grade	Lowest Grade	Totals
Right Eye	Highest Grade	821	112	85	35	1053
	Second Grade	116	494	145	27	782
	Third Grade	72	151	583	87	893
	Lowest Grade	43	34	106	331	514
	Totals	1052	791	919	480	3242

In the table, a total of 3242 individuals are represented. The rows are a categorization of the individuals with respect to their right eye performance in distance vision and the columns are a similar categorization of the individuals with respect to their

left eye performance. By inspection it appears that there is a positive association between the left and right eye ratings. We can test this in a rigorous fashion as follows.

We may regard the table as two rankings of the 3242 individuals, albeit with extensive ties. The rows give a ranking based on right eye performance, with 1053 individuals tied for first, 782 tied for second, 893 tied for third and 514 tied for fourth. The columns give a different ranking of the same 3242 individuals based on left eye performance, with 1052 individuals tied for first, 791 tied for second, 919 tied for third and 480 tied for fourth. The statistical procedure is to calculate the *tau-x* rank correlation between the two rankings and then to test whether the value obtained is statistically significant. The value of *tau-x* for the above table is 0.612, indicating positive association between the ratings for the left and right eyes.

We can test whether the value of *tau-x* obtained from the table is statistically significant using Monte Carlo procedures. According to the null hypothesis of independence between the row and column breakdowns, we may obtain similar tables by randomly assigning each of the 3242 individuals to one of the 16 cells. (The row and column marginal distributions may or may not be fixed.) We do this a large number times and each time we calculate the resulting value of *tau-x*. The proportion of values so obtained which exceed the observed value of *tau-x* gives an estimate of the likelihood of observing such a value purely by chance. In this case we find that for non-fixed marginals, the mean value of *tau-x* after 2000 Monte Carlo iterations is 0.06253 with a standard deviation of 0.01095. The observed value is over 50 standard errors above the mean. None of the 2000 Monte Carlo iterations produced a value as high as the observed value of 0.612. Therefore we may conclude that there is statistically significant positive association between the ratings of the left and right eyes of the tested individuals.

Mathematical Treatment of the 2 by 2 Ordered Table

Let the general 2 by 2 ordered table be given as follows.

a	b	a+b
c	d	c+d
a+c	b+d	n

Constraints: a, b, c, d greater than or equal to 0
n greater than or equal to 2

The constraint that the total number of table entries be greater than or equal to 2 is necessary because if *n* is 1 we have no possibility of different row and column rankings. The formula for calculating *tau-x* for the general 2 by 2 ordered table is given below.

$$\tau_x = \frac{2ad - 2bc + a(a-1) + b(b-1) + c(c-1) + d(d-1)}{n(n-1)}$$

From the formula it can be seen that the value of *tau-x* is 1 when b=c=0 as we would expect since in this case both row and column rankings are the same. Note also that the value of *tau-x* for the 2 by 2 ordered table with a=d=0 is not –1 because although the row and column rankings disagree on the relative order of the sets of b and c individuals, they both agree on the fact that the set of b individuals are tied with each other as are the set of c individuals. In the case where b and c are both equal to 1, the value of *tau-x* is –1.

We may also note that in the case where a=b=c=d, the value of *tau-x* is zero only if n=4 and approaches the value 0.25 asymptotically as *n* grows large. This is in accordance with the fact that both row and column rankings agree with respect to the large number of ties in the 2 by 2 table.

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This report presents a new approach to the problem of finding a solution ranking for the weighted m-rankings problem. The input rankings may be weighted, may contain unlimited ties and may be incomplete. We have developed a new rank correlation coefficient which we have named tau-x. Tau-x is ideal as a measure of agreement in the weighted m-rankings problem. We have developed a branch and bound algorithm to search the solution space and a heuristic algorithm to use in large problems. Our methods are being successfully applied to support high level decision making in the Canadian Department of National Defence.

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